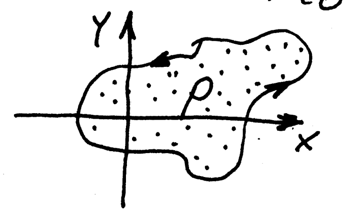
Računanje površine ravne figure

Površinu figure ograničenu zatvorenom linijom c računamo po formuli:

$$P=\frac{1}{2}\int_c x\,dy - y\,dx.$$



Podrazumjeva se da po liniji c prelazimo u pozitivnom smjeru

(#) Pokazati da se površina ograničena jednostavnom zatvorenom krivom (konturom) c računa po formuli

$$\frac{1}{2}\int_c x\,dy - y\,dx$$

Rj. U formuli Greena stavimo $P(x,y)=-y$, $Q(x,y)=x$. Tada

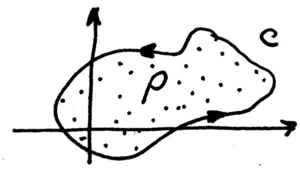
$$\int_c x\,dy - y\,dx = \iint_S \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)dx\,dy = 2\iint_S dx\,dy = 2\cdot P$$

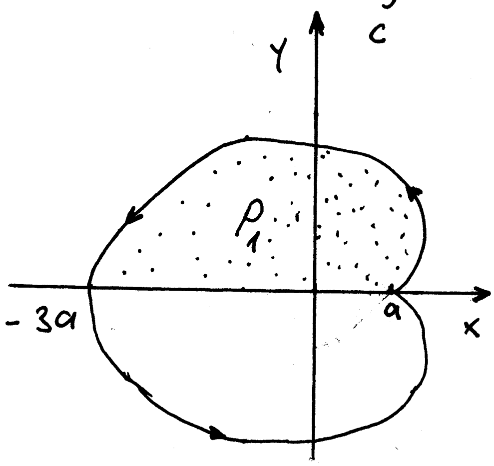
gdje je P tražena površina. Prema tome $P=\frac{1}{2}\int_c x\,dy - y\,dx$

(#) Uz pomoć krivoliniskog integrala druge vrste, izračunati površinu, ograničenu kardioidom $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$.

Rj. Prisjetimo se, površina figure ograničene krivom c se računa po formuli:

$$P = \frac{1}{2}\oint_c x\,dy - y\,dx$$





kardioida
$x = 2a\cos t - a\cos 2t$
$y = 2a\sin t - a\sin 2t$

$t=0$: $x=a$, $y=0$
$t=\pi$: $x=-3a$, $y=0$

Primetimo da je kardioida kriva linija koja je simetrična u odnosu na x-osu, pa da bi izračunali površinu ograničenu kardioidom dovoljno je izračunati površinu iznad x-ose

Da bismo opisali kardioidu parametar t uzima vrijednosti od 0 do 2π.

Prisjetimo se, ako je kriva c data u parametarskom obliku $x = \mu(t)$, $y = \eta(t)$, $t_1 \le t \le t_2$ tada se krivoliniski integral računa po formuli

$$\int_c [P(x,y)\,dx + Q(x,y)\,dy] = \int_{t_1}^{t_2} (P(\mu(t), \eta(t))\mu'(t) + Q(\mu(t), \eta(t))\eta'(t))\,dt$$

$$P = \frac{1}{2}\oint_c x\,dy - y\,dx = \left| \begin{array}{l} x = 2\cos t - \cos 2t \\ dx = (-2\sin t + 2\sin 2t)\,dt \\ y = 2\sin t - \sin 2t \\ dy = (2\cos t - 2\cos 2t)\,dt \end{array} \right| = \frac{1}{2}\int_0^{2\pi} (2\cos t - \cos 2t)\cdot(2\cos t - 2\cos 2t$$

$$= \frac{1}{2}\int_0^{2\pi} \left[(2\cos t - \cos 2t)(2\cos t - 2\cos 2t) - (2\sin t - \sin 2t)(-2\sin t + 2\sin 2t)\right] dt = 2P_1$$

$$= \int_0^{\pi} (\underline{4\cos^2 t} - 6\cos t\cos 2t + \underline{\underline{2\cos^2 2t}} + \underline{4\sin^2 t} - 6\sin t\sin 2t + \underline{\underline{2\sin^2 2t}})\,dt =$$

$$= \int_0^{\pi} (6 - 6\cos t\cos 2t - 6\sin t\sin 2t)\,dt = 6\int_0^{\pi} (1 - \cos(t - 2t))\,dt = \dots = 6\pi$$

(#) Izračunati pomoću krivolinijskog integrala II vrste površinu ravne figure ograničene konturom

$$c: \begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \\ 0 \leq t \leq 2\pi . \end{cases}$$

Rj. Površina figure ograničenu zatvorenom linijom c računamo po formuli $P = \frac{1}{2}\int_c x\,dy - y\,dx$.

$$x = a(t-\sin t) \qquad y = a(1-\cos t)$$

$$dx = a(1-\cos t) \qquad dy = a\sin t$$

$$\begin{aligned} x\,dy - y\,dx &= a(t-\sin t)\cdot a\sin t - a(1-\cos t)\cdot a(1-\cos t) \\ &= a^2 t\sin t - a^2\sin^2 t - a^2(1-\cos t)^2 \\ &= a^2(t\sin t - \sin^2 t - 1 + 2\cos t - \cos^2 t) \\ &= a^2(t\sin t + 2\cos t - 2) \end{aligned}$$

$$P = \frac{1}{2}\int_c x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a^2(t\sin t + 2\cos t - 2))\,dt =$$

$$= \frac{a^2}{2}\left(\int_0^{2\pi} t\sin t\,dt + 2\int_0^{2\pi}\cos t\,dt - 2\int_0^{2\pi}dt\right) = \ldots = \frac{a^2}{2}(-2\pi + 0 - 4\pi) = 3a^2\pi$$

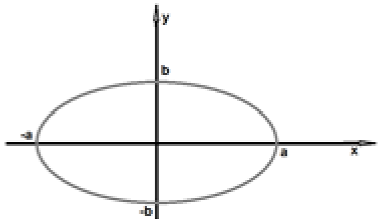
1. Izračunati površinu figure koja je ograničena krivom:

a) elipsom $x = a\,cost,\ \ y = b\,sint$;

b) petljom Dekartovim listom $x^3 + y^3 - 3axy = 0$.

Rješenja:

a)



Slika 1: elipsa

Koristit ćemo sljedeću formulu:

$$P = \frac{1}{2}\oint_{C_1} xdy - ydx,$$

gdje je (vidi sliku 1)

$$C_1 = \begin{cases} x = a\,cost \\ y = b\,sint \\ 0 \le t \le 2\pi \end{cases}$$

Izračunajmo izvode od x i y:

$$dx = -a\,sint\,dt$$

$$dy = b\,cost\,dt$$

Uvrstimo u formulu:

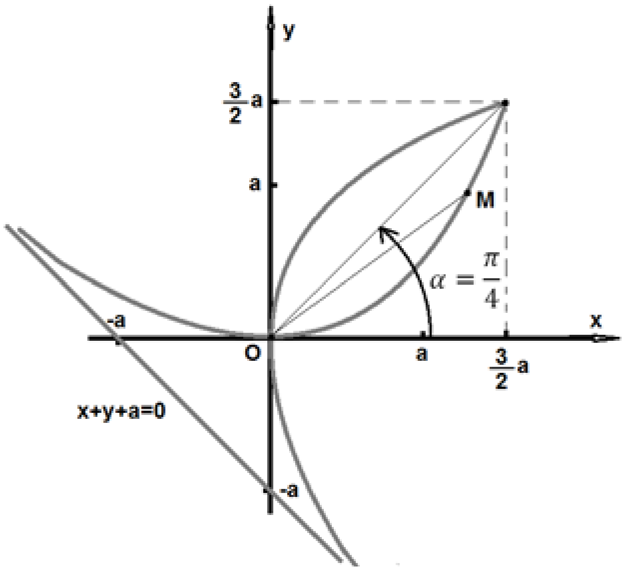
$$P = \frac{1}{2}\oint_{C_1} xdy - ydx = \frac{1}{2}\int_0^{2\pi}\left(a\,cost\,b\,cost - b\,sint\ (-a)\,sint\right)dt$$

$$= \frac{1}{2}\int_0^{2\pi}(ab\,cos^2t + ab\,sin^2t)dt = \frac{1}{2}ab\int_0^{2\pi}\left(cos^2t + sin^2t\right)dt$$

$$= \frac{1}{2}ab\int_0^{2\pi}1\,dt = \frac{1}{2}ab\left(t\Big|_0^{2\pi}\right) = \frac{1}{2}ab\left(2\pi - 0\right) = \boldsymbol{ab\pi}$$

Konačno rješenje: $\boldsymbol{P = ab\pi}$.

b)



Slika 2: Dekartov list

Da bismo koristili formulu

$$P = \frac{1}{2}\oint_{C_1} xdy - ydx,$$

moramo preći na parametarsku jednačinu krive uzevši:

$$y = tx,\ t = \frac{y}{x}$$

Vidimo da polarni radijus OM (vidi sliku 2), gdje je O(0,0) i M(x,y), opisuje cijelu petlju krive kada t ide od 0 do +∞.

Uvrstimo smjenu $y = tx$ u $x^3 + y^3 - 3axy = 0$ te na dobiveni rezultat unijeti i smjenu $x = \frac{y}{t}$

pa ćemo imati:

$$x^3 + (tx)^3 - 3ax(tx) = 0$$

$$x^3\left(1+t^3\right) - 3tax^2 = 0\ /:x^2$$

$$\frac{x^3\ (1+t^3\ ) - 3tax^2}{x^2} = 0$$

$$x\left(1+t^3\right) - 3ta = 0$$

$$x\left(1+t^3\right) = 3ta$$

$$\boldsymbol{x} = \frac{3\boldsymbol{ta}}{1+\boldsymbol{t}^3}$$

$$x = \frac{3ta}{1+t^3}$$

$$\frac{y}{t} = \frac{3ta}{1+t^3}$$

$$\frac{y}{t} = \frac{3ta}{1+t^3}$$

Pa dalje računamo izvod za x:

$$dx = \frac{3a\left(1+t^3\right) - 3ta\ (3t^2)}{(1+t^3)^2} dt$$

$$dx = 3a\frac{1+t^3 - t(3t^2)}{(1+t^3)^2} dt$$

$$\boldsymbol{dx} = 3\boldsymbol{a}\frac{1-2\boldsymbol{t}^3}{(1+\boldsymbol{t}^3)^2}\ \boldsymbol{dt}$$

te i za y :

$$dy = \frac{6at\left(1+t^3\right) - 3at^2(3t^2)}{(1+t^3)^2} dt$$

$$dy = 3at\frac{2\left(1+t^3\right) - t(3t^2)}{(1+t^3)^2} dt$$

$$dy = 3at\frac{2+2t^3 - 3t^3}{(1+t^3)^2} dt$$

$$\boldsymbol{dy} = 3\boldsymbol{at}\frac{2-\boldsymbol{t}^3}{(1+\boldsymbol{t}^3)^2}\ \boldsymbol{dt}$$

Pomnožimo izvode sa dx i dy sa y i x, redom

$$x\,dy = \frac{3ta}{\left(1+t^3\right)} 3at\frac{2-t^3}{(1+t^3)^2} dt$$

$$\boldsymbol{x\ dy} = 9\boldsymbol{a}^2\boldsymbol{t}^2\ \frac{2-\boldsymbol{t}^3}{\left(1+\boldsymbol{t}^3\right)^3}\ \boldsymbol{dt}$$

$$y\,dx = \frac{3t^2a}{1+t^3} 3a\frac{1-2t^3}{(1+t^3)^2} dt$$

$$\boldsymbol{y\ dx} = 9\boldsymbol{a}^2\boldsymbol{t}^2\ \frac{1-2\boldsymbol{t}^3}{(1+\boldsymbol{t}^3)^3}\ \boldsymbol{dt}$$

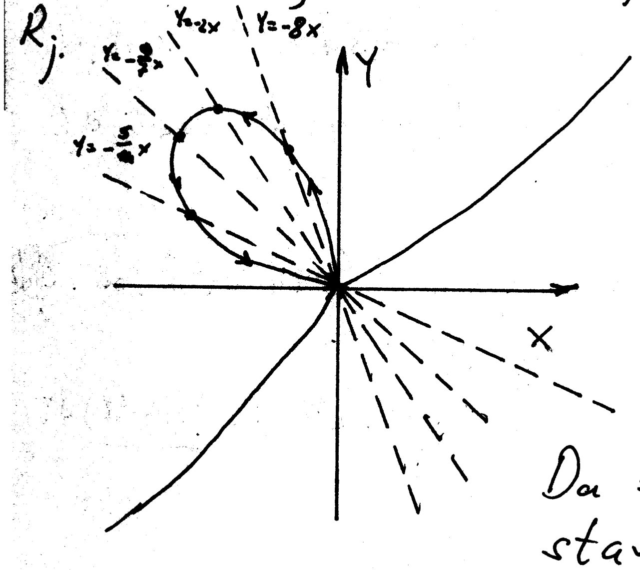
Sad uvrstimo dobijene rezultate:

$$P = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^\infty \left(9a^2t^2\ \frac{2-t^3}{\left(1+t^3\right)^3} - 9a^2t^2\ \frac{1-2t^3}{(1+t^3)^3} \right) dt$$

$$= \frac{1}{2}\int_0^\infty 9a^2t^2\ \frac{2-t^3 -1+2t^3}{\left(1+t^3\right)^3} dt = \frac{9a^2}{2}\int_0^\infty t^2\ \frac{1+t^3}{\left(1+t^3\right)^3} dt =$$

(#) Uz pomoć krivoliniskog integrala izračunati površinu Dekartovog lista dobijen petljom $x^3+y^3-3axy=0$.

Rj.



$$P=\frac{1}{2}\oint_C x\,dy-y\,dx$$

Da bismo upotrebili ovu formulu potrebno je parametrizovati krivu.

Da bismo parametrizovali datu petlju, stavimo $y=tx$. Tada iz jednačine krive dobijamo:

$$x^3+y^3-3axy=0$$
$$x^3+t^3x^3-3atx^2=0 \quad /:x^2$$
$$x(1+t^3)=3at$$
$$x=\frac{3at}{1+t^3}$$

(Pokušajte sa slike shvatiti zašto smo stavili $y=tx$!!!)

$$y=tx$$
$$y=\frac{3at^2}{1+t^3}$$

$$dx=3a\,d\left(\frac{t}{1+t^3}\right)=3a\frac{1+t^3-t\cdot 3t^2}{(1+t^3)^2}dt=3a\frac{1-2t^3}{(1+t^3)^2}dt$$

$$dy=3a\,d\left(\frac{t^2}{1+t^3}\right)=3a\frac{2t(1+t^3)-t^2\cdot 3t^2}{(1+t^3)^2}=3at\frac{2+2t^3-3t^3}{(1+t^3)^2}=3at\frac{2-t^3}{(1+t^3)^2}$$

$$x\,dy=3at\cdot\frac{1}{1+t^2}\cdot 3at\cdot\frac{2-t^3}{(1+t^3)^2}dt=(3at)^2\frac{2-t^3}{(1+t^3)^3}dt$$

$$y\,dx=3at\,\frac{t}{1+t^3}\cdot 3a\frac{1-2t^3}{(1+t^3)^2}dt=(3at)^2\frac{1-2t^3}{(1+t^3)^3}dt$$

$$P=\frac{1}{2}\oint_C x\,dy-y\,dx=\frac{1}{2}\int_{-\infty}^{0}9a^2t^2\frac{\overbrace{2-t^3-1+2t^3}^{1+t^3}}{(1+t^3)^3}dt=\frac{9a^2}{2}\int_{-\infty}^{0}\frac{t^2}{(1+t^3)^2}dt=$$

$$=\left|\begin{array}{l}1+t^3=u\\ 3t^2dt=du\\ t^2dt=\frac{1}{3}du\end{array}\right|=\frac{3a^2}{2}\int_{-\infty}^{0}\frac{du}{u^2}=\frac{3a^2}{2}\cdot\frac{u^{-1}}{-1}=-\frac{3a^2}{2}\cdot\frac{1}{1+t^3}\Bigg|_{-\infty}^{0}$$

$$=-\frac{3a^2}{2}(1-0)=-\frac{3a^2}{2}$$

Površina je uvijek pozitivna $\quad P=\frac{3a^2}{2}$

Izračunati površinu figure koja je ograničena krivom $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \le t \le 2\pi$.

Rj.

$$P = \frac{1}{2}\int_C x\,dy - y\,dx, \qquad C: \begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \\ 0 \le t \le 2\pi \end{cases} \qquad \begin{aligned} dx &= 3a\cos^2 t\cdot(-\sin t)\,dt \\ dy &= 3a\sin^2 t\cos t\,dt \end{aligned}$$

$$P = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}\left(a\cos^3 t\cdot 3a\sin^2 t\cos t - a\sin^3 t\cdot 3a\cos^2 t\cdot(-\sin t)\right)dt$$

$$= \frac{1}{2}\cdot 3a^2\int_0^{2\pi}(\sin^2 t\cos^4 t + \sin^4 t\cos^2 t)\,dt = \frac{3}{2}a^2\int_0^{2\pi}\sin^2 t\cos^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}\,dt$$

$$= \frac{3}{2}a^2\int_0^{2\pi}\frac{1}{4}\underbrace{(2\sin t\cos t)}_{\sin 2t}{}^2\,dt = \frac{3}{8}a^2\int_0^{2\pi}\sin^2 2t\,dt \overset{(*)}{=} \frac{3}{8}a^2\int_0^{2\pi}\frac{1}{2}(1-\cos 4t)\,dt$$

$$= \frac{3}{16}a^2\left(t\Big|_0^{2\pi} - \frac{1}{4}\sin 4t\Big|_0^{2\pi}\right)$$

$$= \frac{3}{16}a^2(2\pi - 0) = \frac{3}{8}a^2\pi$$

$$\begin{aligned} 1 &= \sin^2 2t + \cos^2 2t \\ \cos 4t &= \cos^2 2t - \sin^2 2t \end{aligned} \quad\Rightarrow\quad 1 - \cos 4t = 2\sin^2 2t \quad \ldots(*)$$

Zadaci za vježbu

U zadacima 3861 — 3868 pomoću krivolinijskog integrala izračunati površinu oblasti ograničene datim zatvorenim krivama.

3861. Elipsom $x = a\cos t,\ y = b\sin t$.

3862. Astroidom $x = a\cos^3 t,\ y = a\sin^3 t$.

3863. Kardioidom $x = 2a\cos t - a\cos 2t,\ y = 2a\sin t - a\sin 2t$.

3864*. Petljom dekartova lista $x^3 + y^3 - 3axy = 0$.

3865. Petljom krive $(x+y)^3 = xy$.

3866. Petljom krive $(x+y)^4 = x^2 y$.

3867*. Bernulijevom lemniskatom $(x^2+y^2)^2 = 2a^2(x^2-y^2)$.

3868. Petljom krive $(\sqrt{x}+\sqrt{y})^{12} = xy$.

Rješenja

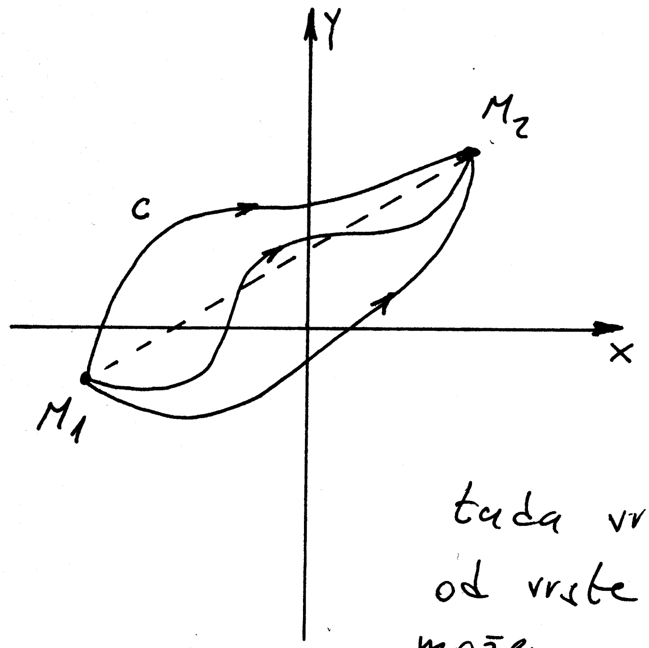
3861. πab. **3862.** $\frac{3}{8}\pi a^2$. **3863.** $6\pi a^2$.

3864*. $\frac{3}{2}a^2$. Preći na parametarske jednačine krive, stavljajući $y = tx$.

3865. $\frac{1}{60}$. **3866.** $\frac{1}{210}$. **3867*.** $2a^2$. Staviti $y = x\,\mathrm{tg}\,t$.

3868*. $\frac{1}{30}$. Staviti $y = xt^2$.

Nezavisnost krivoliniskog integrala od vrste krive linije. Određivanje primitivnih f-ja



Ako je data kriva linija c koja spaja tačke $M_1(a,b)$ i $M_2(c,d)$ (pri čemu je M_1 početak a M_2 kraj krive linije c) i krivoliniski integral $I = \int_C P(x,y)\,dx + Q(x,y)\,dy$

kod kojeg vrijedi $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

tada vrijednost integrala I ne zavisi od vrste krive linije c (za krivu liniju c možemo uzeti bilo koju krivu koja spaja tačke M_1 i M_2).

Vrijednost integrala obično tražimo tako što nađemo f-ju $u = u(x,y)$ za koju vrijedi $du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = P(x,y)dx + Q(x,y)dy$ pa imamo

$$I = \int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C du(x,y) = u(x,y)\Big|_{(a,b)}^{(c,d)} = u(c,d) - u(a,b)$$

Izračunati krivolinijski integral $\int\limits_{(0,1)}^{(3,-4)} x\,dx + y\,dy$.

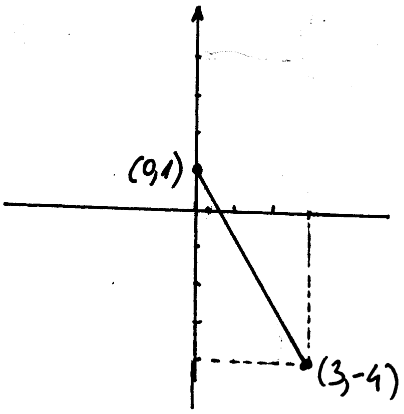
Rj. Integral $I = \int\limits_C P\,dx + Q\,dy$ kod kojeg vrijedi $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, vrijednost integrala I ne zavisi od vrste krive linije C.

U našem slučaju $P(x,y) = x$, $Q(x,y) = y$

$\frac{\partial P}{\partial y} = 0$, $\frac{\partial Q}{\partial x} = 0$

Prema tome vrijednost ne zavisi od vrste izbora krive linije C koja spaja tačke $(0,1)$ i $(3,-4)$.

I način



Ako je c data kriva u ravni opisana jednačinom $y = \eta(x)$ $(x \in [a,b])$ tada

$$\int\limits_C P\,dx + Q\,dy = \int\limits_a^b [P(x,\eta(x)) + Q(x,\eta(x)) \cdot \eta'(x)]\,dx$$

$$y - y_1 = k(x - x_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$A(\overset{x_1}{0}, \overset{y_1}{1})$, $B(3,-4)$

$$y - 1 = \frac{-5}{3}(x - 0)$$

$$y = -\frac{5}{3}x + 1$$

$$y' = -\frac{5}{3}$$

$$\int\limits_{(0,1)}^{(3,-4)} x\,dx + y\,dy = \int\limits_0^3 \left(x + \left(-\frac{5}{3}x + 1\right)\cdot\left(-\frac{5}{3}\right)\right)dx = \int\limits_0^3 \left(x + \frac{25}{9}x - \frac{5}{3}\right)dx =$$

$$= \left(1 + \frac{25}{9}\right)\frac{x^2}{2}\Big|_0^3 - \frac{5}{3}x\Big|_0^3 = \frac{\overset{17}{\cancel{34}}}{\cancel{9}} \cdot \frac{\cancel{9}}{\cancel{2}} - \frac{5}{3}\cdot 3 = 17 - 5 = 12$$

II način

$P(x,y)\,dx + Q(x,y)\,dy = 0$ i $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ je egzaktna diferencijalna jednačina

Rješimo diferenc. jedn. $x\,dx + y\,dy = 0$

$u = u(x,y)$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x} = x \qquad \frac{\partial u}{\partial y} = y \quad \ldots(1)$$

$$\partial u = x\,\partial x$$

$$u = \int x\,dx + \varphi(y) = \frac{1}{2}x^2 + \varphi(y)$$

$$\frac{\partial u}{\partial y} = \varphi'(y) \ldots(2)$$

(1) i (2) $\Rightarrow \varphi'(y) = y$

$$\varphi(y) = \frac{1}{2}y^2$$

$$u = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

Prema tome $u(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$

$$\int_{(0,1)}^{(3,-4)} x\,dx + y\,dy = \int_{(0,1)}^{(3,-4)} du(x,y) = \frac{1}{2}x^2\Big|_{(0,1)}^{(3,-4)} + \frac{1}{2}y^2\Big|_{(0,1)}^{(3,-4)} = \frac{1}{2}(9-0) + \frac{1}{2}(16-1)$$

$$= \frac{9}{2} + \frac{15}{2} = \frac{24}{2} = 12$$

Izračunati integral $\displaystyle\int_{(-2,-1)}^{(3,0)} (x^4+4xy^3)\,dx+(6x^2y^2-5y^4)\,dy$

Rj. Označimo sa $P(x,y)=x^4+4xy^3$ i $Q(x,y)=6x^2y^2-5y^4$

$$\frac{\partial P}{\partial y}=12xy^2 \qquad \frac{\partial Q}{\partial x}=12xy^2$$

$P(x,y)\,dx+Q(x,y)\,dy=0$; $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$ egzaktna diferencijalna jednačina

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x}=P(x,y)=x^4+4xy^3 \qquad \frac{\partial u}{\partial y}=Q(x,y)=6x^2y^2-5y^4 \quad \ldots(*)$$

$$\partial u=P(x,y)\,\partial x$$

$$u=\int(x^4+4xy^3)\,dx=\frac{1}{5}x^5+4\cdot\frac{1}{2}x^2y^3+\varphi(y)=\frac{1}{5}x^5+2x^2y^3+\varphi(y)$$

$$\frac{\partial u}{\partial y}=6x^2y^2+\varphi'(y)\ldots(**) \qquad \text{Iz } (*) \text{ i } (**) \Rightarrow \varphi'(y)=-5y^4$$

$$\varphi(y)=-5\int y^4dy=-y^5$$

Prema tome $u(x,y)=\frac{1}{5}x^5+2x^2y^3-y^5$

$$\int_{(-3-1)}^{(3,0)}(x^4+4xy^3)\,dx+(6x^2y^5-5y^4)\,dy=\int_{(-2,1)}^{(3,0)}d\,u(x,y)=\left(\frac{1}{5}x^5+2x^2y^3-y^5\right)\Bigg|_{(-3,1)}^{(3,0)}$$

$$=\left(\frac{3^5}{5}+0+0\right)-\left(\frac{(-2)^5}{5}+2\cdot4-1\right)=\frac{243}{5}+\frac{32}{5}-\frac{40}{5}+\frac{5}{5}=\frac{240}{5}$$

$$=48$$

(#) Dokazati da integral $\int\limits_L f(xy)(y\,dx + x\,dy)$ po zatvorenoj konturi L ima vrijednost 0 (nula) bez obzira na tip f-je uključen u integrand.

Rj. $\int\limits_L f(xy)(y\,dx + x\,dy) = \int\limits_L y f(xy)\,dx + x f(xy)\,dy$

Označimo sa $P(x,y) = y f(xy)$ i $Q(x,y) = x f(xy)$. Imamo

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= f(xy) + y \cdot \frac{\partial f}{\partial (xy)} \cdot x = f(xy) + xy \cdot \frac{\partial f}{\partial (xy)} \\ \frac{\partial Q}{\partial x} &= f(xy) + x \frac{\partial f}{\partial (xy)} \cdot y = f(xy) + xy \frac{\partial f}{\partial (xy)} \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\int\limits_C P\,dx + Q\,dy = \iint\limits_S \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy \quad \text{formula Greena}$$

$$\int\limits_L f(xy)(y\,dx + x\,dy) = \iint\limits_S \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = 0 \quad \text{bez obzira na } L.$$

q.e.d.

(#) Izračunati krivolinijski integral $\int\limits_C \cos 2y\,dx - 2x\sin 2y\,dy$

gdje je c neka kriva koja spaja tačke $A(1, \frac{\pi}{6})$ i $B(2, \frac{\pi}{4})$.

Rj. Označimo sa $P(x,y) = \cos 2y$ i $Q(x,y) = (-2x)\sin y$

$$\frac{\partial P}{\partial y} = -2\sin 2y \qquad \frac{\partial Q}{\partial x} = -2\sin y \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

vrijednost integrala ne zavisi od vrste konture

I način:

$$\int\limits_C P(x,y)\,dx + Q(x,y)\,dy = \int\limits_C du(x,y) = u(x,y)\Big|_{(a,b)}^{(c,d)}$$ gdje je

$du(x,y) = P(x,y)\,dx + Q(x,y)\,dy$, tačka (a,b) početak a (c,d) kraj konture c

Odredimo f-ju $u = u(x,y)$

$$\frac{\partial u}{\partial x} = P(x,y) = \cos 2y, \qquad \frac{\partial u}{\partial y} = Q(x,y) = -2x\sin 2y$$

$P(x,y)\,dx + Q(x,y)\,dy = 0$; $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ovo je egzaktna diferencijalna jednačina

$$\frac{\partial u}{\partial x} = \cos 2y$$

$$\partial u = \cos 2y\,\partial x \qquad\qquad \frac{\partial u}{\partial y} = -2x\sin 2y \quad \dots (*)$$

$$u = \int \cos 2y\,dx = x\cos 2y + \varphi(y) \Rightarrow \frac{\partial u}{\partial y} = x\cdot(-\sin 2y)\cdot 2 + \varphi'(y) =$$

$$= -2x\sin 2y + \varphi'(y) \quad \dots (**)$$

Sad imamo $\overset{\text{iz } (*) \text{ i } (**)}{\varphi'(y) = 0} \Rightarrow \varphi(y) = c$

$$u(x,y) = x\cos 2y + c$$

$$\int\limits_C \cos 2y\,dx - 2x\sin 2y\,dy = \int\limits_C d(x\cos 2y + c) = x\cos 2y\Big|_{(1,\frac{\pi}{6})}^{(2,\frac{\pi}{4})} + c\Big|_{(1,\frac{\pi}{6})}^{(2,\frac{\pi}{4})} =$$

$$= 2\cos\frac{\pi}{2} - \cos\frac{\pi}{3} + (c - c) = -\frac{1}{2}$$

II način: (za vježbu) standardno rješavamo krivolinijski integral s tim da izaberemo pogodnu konturu koja spaja date tačke

(#) Izračunati integral po glatkom luku koji spaja tačke A i B

$$\int_{\widehat{AB}} \left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}dz$$

$A(1,1,1)$, $B(1,2,3)$, $\widehat{AB} \subseteq \{(x,y,z) \mid x>0,\ y>0,\ z>0\}$.

Rj. Označimo sa $P(x,y,z)=1-\frac{1}{y}+\frac{y}{z}$, $Q(x,y,z)=\frac{x}{z}+\frac{x^2}{y^2}$,

$R(x,y,z)=-\frac{xy}{z^2}$, i izračunajmo $\frac{\partial^2 P}{\partial y \partial z}$, $\frac{\partial^2 Q}{\partial x \partial z}$ i $\frac{\partial^2 R}{\partial x \partial y}$

$$\frac{\partial P}{\partial y}=-(-1)y^{-2}+\frac{1}{z} \qquad \frac{\partial Q}{\partial x}=\frac{1}{z}+\frac{1}{y^2} \qquad \frac{\partial R}{\partial x}=-\frac{y}{z^2}$$

$$\frac{\partial^2 P}{\partial y \partial z}=-\frac{1}{z^2} \qquad \frac{\partial^2 Q}{\partial x \partial z}=-\frac{1}{z^2} \qquad \frac{\partial^2 R}{\partial x \partial y}=-\frac{1}{z^2}$$

Kako je $\frac{\partial^2 P}{\partial y \partial z}=\frac{\partial^2 Q}{\partial x \partial z}=\frac{\partial^2 R}{\partial x \partial y}$ to integral ne zavisi od vrste krive linije koja spaja tačke A i B.

Odredimo f-ju $u=u(x,y,z)$ za koju vrijedi da je

$$du=\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}dz$$

$$\frac{\partial u}{\partial x}=1-\frac{1}{y}+\frac{y}{z}$$

$$u=\int\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\varphi(y,z)$$

$$u=x-\frac{x}{y}+\frac{xy}{z}+\varphi(y,z)$$

$$\frac{\partial u}{\partial y}=\frac{x}{y^2}+\frac{x}{z}+\varphi'_y(y,z)$$

$$\frac{\partial u}{\partial y}=\frac{x}{y^2}+\frac{x}{z}$$

$$\varphi'_y(y,z)=0$$

$$\varphi(y,z)=C+\psi(z) \quad \dots(1)$$

$$\frac{\partial u}{\partial z}=-\frac{xy}{z^2}+\varphi'_z$$

$$\frac{\partial u}{\partial z}=-\frac{xy}{z^2}$$

$$\varphi'_z=0 \quad \dots(2)$$

(1) i (2) $\Rightarrow$ $\psi(z)=0$ $\Rightarrow$ $\varphi(y,z)=C$

$$u=x-\frac{x}{y}+\frac{xy}{z}+C$$

traženo rješenje

$$\int_{\widehat{AB}}\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}dz=\int_{\widehat{AB}}du=\left(x-\frac{x}{y}+\frac{xy}{z}\right)\Bigg|_{(1,1,1)}^{(1,2,3)}=1-\frac{1}{2}+\frac{2}{3}-1=\frac{1}{6}$$

(#) Izračunati krivoliniski integral $\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2}$ duž putanje koja ne siječe osu Oy.

Rj.

Vrijednost integrala $I = \int_c P(x,y)\,dx + Q(x,y)\,dy$ ne zavisi od vrste konture c ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

U našem slučaju $I = \int_{(2,1)}^{(1,2)} \frac{y}{x^2}\,dx - \frac{1}{x}\,dy$ $\quad P(x,y) = \frac{y}{x^2}, \quad Q(x,y) = -\frac{1}{x}$

$$\frac{\partial P}{\partial y} = \frac{1}{x^2}, \quad \frac{\partial Q}{\partial x} = \frac{1}{x^2}$$

Prema tome vrijednost integrala ne zavisi od vrste krive linije c koja spaja tačke $(2,1)$ i $(1,2)$.

I način: Odredićemo primitivnu f-ju

$$u = u(x,y)$$

$P(x,y)\,dx + Q(x,y)\,dy = 0$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ovo je egzaktna dif. jednačina

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$du = \frac{y}{x^2}\,dx - \frac{1}{x}\,dy$$

$\partial u = \frac{y}{x^2}\,\partial x$ $\qquad \frac{\partial u}{\partial x} = \frac{y}{x^2}, \quad \frac{\partial u}{\partial y} = -\frac{1}{x} \quad \ldots(1)$

$$u = \int \frac{y}{x^2}\,dx + \varphi(y) = y\,\frac{x^{-1}}{-1} + \varphi(y) = -\frac{y}{x} + \varphi(y)$$

$\frac{\partial u}{\partial y} = -\frac{1}{x} + \varphi'(y) \quad \ldots(2)$ $\qquad (1)$ i (2) $\Rightarrow \varphi'(y) = 0$

$$\varphi(y) = c$$

$$u = -\frac{y}{x} + c$$

$$\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2} = \int_{(2,1)}^{(1,2)} du = -\frac{y}{x}\Big|_{(2,1)}^{(1,2)} = -\frac{2}{1} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 2 = -\frac{3}{2}$$

II način: Spojimo tačke $(2,1)$ i $(1,2)$ nekom krivom (ili pravom) ili izlomljenom pravom linijom i izračunamo integral na klasičan način.

Izračunati krivoliniski integral $\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$ duž puta

koji ne prolazi kroz koordinatni početak.

Rj: Ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ tada vrijednost integrala $\int_C P dx + Q dy$ ne zavisi od vrste izbora puta integracije.

$$I = \int_{(1,0)}^{(6,8)} \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy \Rightarrow \left.\begin{array}{l} P(x,y) = \frac{x}{\sqrt{x^2+y^2}} \\ Q(x,y) = \frac{y}{\sqrt{x^2+y^2}} \end{array}\right\} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -\frac{xy}{\sqrt{(x^2+y^2)^3}}$$

Prema tome vrijednost integrala ne zavisi od izbora krive kojom ćemo spojiti tačke $(1,0)$ i $(6,8)$.

I način: Odrediti ćemo primitivnu f-ju u.

$u = u(x,y)$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$du = \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy$$

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \quad \ldots(1)$$

$$\partial u = \frac{x}{\sqrt{x^2+y^2}} \partial x$$

$$u = \int \frac{x}{\sqrt{x^2+y^2}} dx + \varphi(y) = \left| \begin{array}{c} x^2+y^2 = t^2 \\ 2x\,dx = 2t\,dt \\ x\,dx = t\,dt \end{array} \right| = \int \frac{t}{\sqrt{t^2}} dt + \varphi(y)$$

$$= t + \varphi(y) = \sqrt{x^2+y^2} + \varphi(y)$$

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} + \varphi'(y) \quad \ldots(2)$$

(1) i (2) $\Rightarrow \varphi'(y) = 0 \Rightarrow u = \sqrt{x^2+y^2}$

$$\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \int_{(1,0)}^{(6,8)} du = u\Big|_{(1,0)}^{(6,8)} = \sqrt{x^2+y^2}\Big|_{(1,0)}^{(6,8)} = \sqrt{36+64} - \sqrt{1+0} = 9$$

II način: Spojimo tačke $(1,0)$ i $(6,8)$ nekom krivom koja ne prolazi kroz koordinatni početak i izračunamo integral na klasičan način.

Zadaci za vježbu

U zadacima 3831 — 3835 uveriti se da su vrednosti datih integrala, uzetih po zatvorenim konturama, jednake nuli bez obzira na oblik funkcija koje ulaze u podintegralni izraz.

3831. $\int\limits_L \varphi(x)\,dx+\psi(y)\,dy.$ **3832.** $\int\limits_L f(xy)\,(y\,dx+x\,dy).$

3833. $\int\limits_L f\left(\frac{y}{x}\right)\frac{x\,dy-y\,dx}{x^2}.$

3834. $\int\limits_L [f(x+y)+f(x-y)]\,dx+[f(x+y)-f(x-y)]\,dy.$

3835. $\int\limits_L f(x^2+y^2+z^2)\,(x\,dx+y\,dy+z\,dz).$

3836*. Dokazati da integral $\int\limits_L \frac{x\,dy-y\,dx}{x+y}$, uzet u pozitivnom smeru obilaženja po bilo kojoj zatvorenoj konturi koja obuhvata koordinatni početak, ima vrednost 2π.

3837. Izračunati $\int\limits_L \frac{x\,dy-y\,dx}{x^2+4y^2}$ duž kruga $x^2+y^2=1$ u pozitivnom smeru obilaženja.

U zadacima 3838—3844 izračunati krivolinijske integrale totalnih diferencijala.

3838. $\int\limits_{(-1,\,2)}^{(2,\,3)} y\,dx+x\,dy.$ **3839.** $\int\limits_{(0,\,0)}^{(2,\,1)} 2xy\,dx+x^2\,dy.$

3840. $\int\limits_{(3,\,4)}^{(5,\,12)} \frac{x\,dx+y\,dy}{x^2+y^2}$ (koordinatni početak ne leži na putanji integracije).

3841. $\int\limits_{(P_1)}^{(P_2)} \frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}$, pri čemu tačke P_1 i P_2 leže na koncentričnim krugovima čiji je zajednički centar u koordinatnom početku, a poluprečnici su im R_1 i R_2 (koordinatni početak ne leži na putanji integracije).

3842. $\int\limits_{(1,\,-1,\,2)}^{(2,\,1,\,3)} x\,dx-y^2\,dy+z\,dz.$

3843. $\int\limits_{(1,\,2,\,3)}^{(3,\,2,\,1)} yz\,dx+zx\,dy+xy\,dz.$

3844. $\int\limits_{(7,\,2,\,3)}^{(5,\,3,\,1)} \frac{zx\,dy+xy\,dz-yz\,dx}{(x-yz)^2}$ $\left(\text{putanja integracije ne preseca površinu } z=\frac{x}{y}\right).$

U zadacima 3845—3852 naći funkcije čiji su totalni diferencijali zadati.

3845. $du=x^2\,dx+y^2\,dy.$

3846. $du=4(x^2-y^2)(x\,dx-y\,dy).$

3847. $du=\frac{(x+2y)\,dx+y\,dy}{(x+y)^2}.$

3848. $du=\frac{x}{y\sqrt{x^2+y^2}}\,dx-\left(\frac{x^2+\sqrt{x^2+y^2}}{y^2\sqrt{x^2+y^2}}\right)dy.$

3849. $du=\left[\frac{x-2y}{(y-x)^2}+x\right]dx+\left[\frac{y}{(y-x)^2}-y^2\right]dy.$

3850. $du=(2x\cos y-y^2\sin x)\,dx+(2y\cos x-x^2\sin y)\,dy.$

3851. $du=\frac{2x(1-e^y)}{(1+x^2)^2}\,dx+\left(\frac{e^y}{1+x^2}+1\right)dy.$

Rješenja

3836*. Primeniti Grinovu formulu na dvostruko povezanu oblast, ograničenu zatvorenom konturom L i bilo kakvim krugom čiji je centar u koordinatnom početku i koji ne preseca konturu L.

3837. π. **3838.** 8.

3839. 4. **3840.** $\ln\frac{13}{5}$.

3841. R_2-R_1. **3842.** $\frac{10}{3}$.

3843. 0. **3844.** $-\frac{9}{2}$.

3845. $u=\frac{x^3+y^3}{3}+C.$

3846. $u=(x^2-y^2)^2+C.$

3847. $u=\ln|x+y|-\frac{y}{x+y}+C.$

3848. $u=\frac{\sqrt{x^2+y^2}+1}{y}+C.$

3849. $u=\ln|x-y|+\frac{y}{x-y}+\frac{x^2}{2}-\frac{y^3}{3}+C.$

3850. $u=x^2\cos y+y^2\cos x+C.$

3851. $u=\frac{e^y-1}{1+x^2}+y+C.$

3852. $du=\dfrac{(3y-x)\,dx+(y-3x)\,dy}{(x+y)^3}$.

3853. Odrediti broj n tako da izraz $\dfrac{(x-y)\,dx+(x+y)\,dy}{(x^2+y^2)^n}$ bude totalni diferencijal, i naći odgovarajuću primitivnu funkciju.

3854. Odrediti konstante a i b tako da izraz

$$\frac{(y^2+2xy+ax^2)\,dx-(x^2+2xy+by^2)\,dy}{(x^2+y^2)^2}$$

bude totalan diferencijal, i naći odgovarajuću primitivnu funkciju.

U zadacima 3855 — 3860 naći funkcije čiji su totalni diferencijali zadati.

3855. $du=\dfrac{dx+dy+dz}{x+y+z}$. **3856.** $du=\dfrac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2}}$.

3857. $du=\dfrac{yz\,dx+xz\,dy+xy\,dz}{1+x^2y^2z^2}$.

3858. $du=\dfrac{2\,(zx\,dy+xy\,dz-yz\,dx)}{(x-yz)^2}$.

3859. $du=\dfrac{dx-3\,dy}{z}+\dfrac{3y-x+z^3}{z^2}\,dz$.

3860. $du=e^{\frac{y}{z}}dx+\left(\dfrac{e^{\frac{y}{z}}(x+1)}{z}+ze^{yz}\right)dy+\left(-\dfrac{e^{\frac{y}{z}}(x+1)\,y}{z^2}+ye^{yz}+e^{-z}\right)dz$.

Rješenja

3852. $u=\dfrac{x-y}{(x+y)^2}+C$. **3853.** $n=1$, $u=\dfrac{1}{2}\ln(x^2+y^2)+\operatorname{arctg}\dfrac{y}{x}+C$.

3854. $a=b=-1$, $u=\dfrac{x-y}{x^2+y^2}+C$. **3855.** $u=\ln|x+y+z|+C$.

3856. $u=\sqrt{x^2+y^2+z^2}+C$. **3857.** $u=\operatorname{arctg}xyz+C$.

3858. $u=\dfrac{2x}{x-yz}+C$. **3859.** $u=\dfrac{x-3y}{z}+\dfrac{z^2}{2}+C$.

3860. $u=e^{\frac{y}{z}}(x+1)+e^{yz}-e^{-z}$.